LEBANESE AMERICAN UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICS

MTH 201 - CALCULUS 3

EXAM 1 – FALL 2011

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Duration: 75 minutes

Name: Solutions **ID#:**

- **This exam consists of 9 pages and 8 problems.**
- **Answer the questions below on the space provided. You can use the back pages for scratch or for more space for your answers. Please specify.**
- **Make sure you justify all your answers.**

Question Number	Grade
1. 10%	
2. 12%	
3. 20%	
4. 12%	
5. 12%	
6. 10%	
7. 12%	
8. 12%	
TOTAL	

Problem 1: (10%) Evaluate the following limits

(a) $\lim_{x\to\infty} \sin^{-1}(\frac{x^2}{1+2x^2})$ $\approx \sin^{-1}\left(\frac{1}{2}\right) = \pi/6$

5

(b) $\lim_{x\to\infty} \frac{\tan^{-1}(e^x)}{e^{2x}+x}$ $\approx \frac{\tan^{-1}(e^x)}{e^{2x}} = \frac{\pi/2}{\infty} \to 0$

5

Problem 2: (12%)

(a) Simplify the following expression

$$\ln(\cosh 6x - \sinh 6x) + \ln(\cosh 3x + \sinh 3x)$$

6

$$\ln\left(\frac{e^{6x}+e^{-6x}}{2} - \frac{e^{6x}-e^{-6x}}{2}\right) + \ln\left(\frac{e^{3x}+e^{-3x}}{2} + \frac{e^{3x}-e^{-3x}}{2}\right)$$

$$= \ln\left(e^{-6x}\right) + \ln\left(e^{3x}\right) = -6x + 3x = -3x$$

(b) Evaluate

$$\int \frac{\sinh(\ln x)}{x}\,dx$$

Simplify your answer.

6

$$= \int \frac{e^{\ln x} - e^{-\ln x}}{2x}\,dx$$

$$= \int \frac{x - 1/x}{2x}\,dx = \int \frac{1}{2} - \frac{1}{2x^2}\,dx$$

$$= \frac{x}{2} + \frac{1}{2x} + C$$

Method 2

OR

$$\int \sinh u\,du = \cosh u + C = \cosh(\ln x) + C$$

$$= \frac{e^{\ln x} + e^{-\ln x}}{2} + C$$

$$= \frac{x + 1/x}{2} + C$$

6%

Problem 4: (12%) Evaluate the following improper integrals

(a) $\int_2^4 \frac{1}{x^2-x-2}\,dx$ $= \int \frac{dx}{(x-2)(x+1)} = \int \frac{1/3}{x-2} + \frac{-1/3}{x+1}$

$= \frac{1}{3} \ln \left| \frac{x-2}{x+1} \right|$

$= \lim_{t \to 2^+} \int_t^4 \frac{dx}{x^2-x-2} = \lim_{t \to 2^+} \frac{1}{3} \left[\ln\left|\frac{t-2}{t+1}\right| - \ln\left|\frac{2}{5}\right| \right]$

$\to \ln 0 - \ln \frac{2}{5}$

$\to -\infty$

$\longrightarrow -\infty$

$\Rightarrow$ Diverges !!

6%

(b) $\int_{-\infty}^{\infty} \frac{1}{(e^x+e^{-x})}\,dx$: even fn

$\Rightarrow$ Consider: $\int_0^\infty \frac{1}{e^x+e^{-x}}\,dx \approx \int_0^\infty \frac{dx}{e^x}$: converges to $\underline{\underline{L}}$.

$\therefore \int_{-\infty}^{0} \frac{dx}{e^x+e^{-x}}$ converges to $\underline{\underline{+L}}$

$\therefore$ the whole int. converges to $L + L = 2L$

Find $L = \int_0^\infty \frac{e^x}{e^{2x}+1}\,dx = \lim_{t\to\infty} \int \frac{du}{u^2+1}$

5

$= \tan^{-1} e^x \Big|_0^\infty = \tan^{-1}\infty - \tan^{-1} 1$

$= \pi/2 - \pi/4$

$= \pi/4$

$2L = \pi/2$

Problem 5: (12%) Determine the convergence or divergence of the following improper integrals. Justify your answers.

(a) $\int_0^\infty \frac{1}{(e^x+1)^2}\,dx$ $\approx \int_0^1 \frac{dx}{(e^x+1)^2} + \int_1^\infty \frac{dx}{e^{2x}}$

6%

Bounded

$\frac{1}{e^{2x}} < \frac{1}{x^4} \therefore \int_1^\infty \frac{dx}{e^{2x}}$ conv. by DCT with $\int \frac{dx}{x^4}$

$\therefore$ the whole int. converges

(b) $\int_1^\infty \frac{\ln x}{e^x}\,dx$

6%

$\ln x < x$ $\because$

$e^x > x^{10} \therefore \frac{1}{e^x} < \frac{1}{x^{10}}$

$\therefore \frac{\ln x}{e^x} < \frac{x}{x^{10}} = \frac{1}{x^9}$ $\because$ Since $\int_1^\infty \frac{dx}{x^9}$ converges p-int $p>1$

$\therefore \int_1^\infty \frac{\ln x}{e^x}\,dx$ conv. by DCT.

Problem 6: (10%) Show that

$$\int_1^\infty \frac{\sin x + 2}{x^2}\, dx$$

converges, whereas

$$\int_1^\infty \frac{\sin x + 2}{x}\, dx$$

diverges.

$-1 \leq \sin x \leq 1$

$1 \leq \sin x + 2 \leq 3 \quad \therefore \quad \frac{\sin x + 2}{x^2} \leq \frac{3}{x^2}$

Since $\int_1^\infty \frac{3}{x^2}\, dx$ conv. (p-int. $p > 1$)

$\therefore \int_1^\infty \frac{\sin x + 2}{x^2}\, dx$ conv. by DC,

II $\quad \frac{\sin x + 2}{x} \geq \frac{1}{x} \quad \therefore$ Since $\int_1^\infty \frac{dx}{x}$ div.

it follows that $\int_1^\infty \frac{\sin x + 2}{x}\, dx$ div. by DC,

Problem 7: (12%)

Find the values of p for which

$$\int_1^\infty \frac{x}{\sqrt{x^p+1}}\,dx \approx \int_1^\infty \frac{x}{x^{p/2}}\,dx$$

converges.

$$= \int_1^\infty \frac{1}{x^{p/2-1}}\,dx$$

Justify your answer.

$\therefore$ We need $\frac{p}{2} - 1 > 1$ for the int. to conv

$\Rightarrow$ need $\frac{p}{2} > 2$ $\boxed{p > 4}$

Problem 8: (12%) Determine if the following sequences converge or diverge. Justify your answers.

(a) $a_n = \dfrac{n \sin\left((2n-1)\frac{\pi}{2}\right)}{n+1}$ $\approx$ $\sin(2n-1)\pi/2 \to -1$

6

6

(b) $a_n = (1+\frac{2}{n})^n \frac{1}{\sqrt[n]{n^2}}$ $\longrightarrow \dfrac{e^2}{n^{2/n}} \longrightarrow \dfrac{e^2}{1} \longrightarrow e^2$

since $n^{1/n} \to 1$ and so $n^{2/n} \to 1$